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THE EFFICIENCY OF THREE TESTS FOR LINEAR
HYPOTHESES CONCERNING TWO PROBABILITIES

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The efficiency of three tests for linear hypotheses concerning two probabilities^{*)}

by J.M. Buhrman

SUMMARY

Two experiments have probabilities of success p_1 and p_2 respectively. Three methods are described to test hypotheses with the form $p_1 - cp_2 = d$, and their Pitman-efficiencies are given.

KEY WORDS & PHRASES: *Bernoulli trials, linear hypotheses, comparison of probabilities, Pitman efficiency.*

^{*)}

This report will be submitted for publication elsewhere.

1. DESCRIPTION OF THE TESTS

Two experiments A and B have probabilities of success p_1 and p_2 respectively. We consider the general hypotheses

$$(1.1) \quad p_1 - cp_2 = d$$

where c and d are given numbers satisfying the conditions $c > 0$ and $-c < d < 1$. Our aim is a comparison of the following three methods for testing hypotheses of type (1.1).

- I Fixed numbers n_1 and n_2 of experiments A and B respectively are performed. If $\underline{s}_1$ and $\underline{s}_2$ are the respective numbers of successes, then under the hypotheses $p_1 - p_2 = 0$ the statistic $\underline{s}_1 | \underline{s}_1 + \underline{s}_2 = r$ (i.e. $\underline{s}_1$ under the condition $\underline{s}_1 + \underline{s}_2 = r$) has a hypergeometric- (n_1+n_2, n_1, r) distribution. So $(\underline{s}_1, \underline{s}_2)$ provides a test for testing (1.1) with $c = 1$ and $d = 0$ (Fisher-Yates-Irwin test).
- II Random numbers $\underline{n}_1$ and $\underline{n}_2$ experiments A and B respectively are performed. $\underline{n}_1$ is binomial- (n, π) and $\underline{n}_2 \stackrel{\text{def}}{=} n - \underline{n}_1$. Under the hypothesis $p_1 - cp_2 = 0$ the statistic $\underline{s}_1 | \underline{s}_1 + \underline{s}_2 = r$ has a binomial- $(r, \pi c / (\pi c + 1 - \pi))$ distribution, so the statistic $(\underline{s}_1, \underline{s}_2)$ provides a test for the hypothesis (1.1) with $c > 0$ and $d = 0$.
- III If, in the experimental design of II, one chooses $\pi = (1+c)^{-1}$, then under the hypothesis $p_1 - cp_2 = d$ the statistic $\underline{s}_1 + \underline{f}_2$ ($\underline{f}_1$ and $\underline{f}_2$ denote the respective numbers of failures) has a binomial- $(n, (d+c)/(1+c))$ distribution, so it provides a test for (1.1) with $c > 0$ and $-c < d < 1$.

Method I is well-known and frequently discussed (see e.g. [2], [8]), methods II and III are described in [3], [4] and [5].

2. THE PITMAN-EFFICIENCIES

For each method we shall give the asymptotic power of the one-sided test under a sequence of alternatives which converges with order $O(n^{-\frac{1}{2}})$ to the hypothesis as the total number, n , of experiments tends to infinity. Throughout the remainder of this report Φ will denote the standard normal

distribution function, and u_α is defined by $\Phi(u_\alpha) = 1 - \alpha$. Further p_1 and p_2 will be the probabilities of success of an experiment of type A and an experiment of type B respectively.

First some general lemmas are given. Let $V_\theta(\underline{z})$ denote the distribution of a random variable $\underline{z}$ with respect to a probability measure indicated by its parameter θ .

LEMMA 2.1. Let $V_p(\underline{z}_n) = \text{bin}-(n, p)$ and let $\theta_n \rightarrow \theta$. Then

$$(2.1) \quad V_{\theta_n} \left(\frac{\underline{z}_n - n\theta_n}{\sqrt{n}} \right) \rightarrow N(0, \sqrt{\theta(1-\theta)})$$

PROOF. $\underline{z}_n^* \stackrel{\text{def}}{=} (\underline{z}_n - n\theta_n) / \sqrt{n\theta_n(1-\theta_n)}$ is asymptotically standard normal, the Lindeberg-Feller condition being satisfied. $\sqrt{\theta_n(1-\theta_n)} \rightarrow \sqrt{\theta(1-\theta)}$ proves the lemma. $\square$

LEMMA 2.2. Let $V_p(\underline{z}) = \text{bin}-(n, p)$ and let k_n denote the critical value for testing $p = \theta$ against $p > \theta$ at level α . Let $\theta_n = \theta + b/\sqrt{n} + o(1/\sqrt{n})$, then

$$(2.2) \quad P_{\theta_n}(\underline{z}_n \geq k_n) \rightarrow 1 - \Phi(u_\alpha - b/\sqrt{\theta(1-\theta)})$$

PROOF.

$$P_{\theta}(\underline{z}_n \geq k_n) \rightarrow \alpha \quad \Rightarrow \quad \frac{k_n - n\theta}{\sqrt{n\theta(1-\theta)}} \rightarrow u_\alpha \quad (\text{see e.g. [11], Satz 2.11})$$

$$\begin{aligned} P_{\theta_n}(\underline{z}_n \geq k_n) &= P_{\theta_n} \left(\frac{\underline{z}_n - n\theta_n}{\sqrt{n\theta_n(1-\theta_n)}} \geq \frac{k_n - n\theta_n}{\sqrt{n\theta_n(1-\theta_n)}} - \frac{b}{\sqrt{\theta(1-\theta)}} + o(1) \right) \rightarrow \\ &\rightarrow 1 - \Phi(u_\alpha - b/\sqrt{\theta(1-\theta)}) \quad \square \end{aligned}$$

LEMMA 2.3. Let $(\underline{a}_n, \underline{r}_n)$ be a statistic for testing H_n against K_n at level α_n ($n = 1, 2, \dots$), with for $\underline{r}_n = r$ a conditional critical region of the form $\{(a, r): a \geq k_{n,r}\}$. Let

$$\underline{v}_n \stackrel{\text{def}}{=} \underline{a}_n - E(\underline{a}_n | \underline{r}_n; H_n)$$

$$k_{n,r}^* \stackrel{\text{def}}{=} \frac{k_{n,r} - E(a_{\underline{n}} | \underline{r}_{\underline{n}} = r; H_{\underline{n}})}{\sigma(a_{\underline{n}} | \underline{r}_{\underline{n}} = r; H_{\underline{n}})}$$

If

$$(2.3) \quad k_{n,r_{\underline{n}}}^* \rightarrow u_{\alpha} \text{ in probability under } \{K_{\underline{n}}\}$$

$$(2.4) \quad \frac{\sigma(a_{\underline{n}} | \underline{r}_{\underline{n}}; H_{\underline{n}})}{\sigma(\underline{v}_{\underline{n}} | K_{\underline{n}})} \rightarrow 1 \text{ in probability under } \{K_{\underline{n}}\}$$

$$(2.5) \quad \underline{v}_{\underline{n}} \text{ is asymptotically normal under } \{K_{\underline{n}}\}$$

$$(2.6) \quad \eta \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{E(\underline{v}_{\underline{n}} | K_{\underline{n}})}{\sigma(\underline{v}_{\underline{n}} | K_{\underline{n}})} \text{ exists}$$

then the power $\beta_n(K_{\underline{n}})$ satisfies

$$(2.7) \quad \lim_{n \rightarrow \infty} \beta_n(K_{\underline{n}}) = 1 - \Phi(u_{\alpha} - \eta)$$

PROOF.

$$\begin{aligned} \beta_n(K_{\underline{n}}) &= P(a_{\underline{n}} \geq k_{n,r_{\underline{n}}}^* | K_{\underline{n}}) = \\ &= P(\underline{v}_{\underline{n}} \geq k_{n,r_{\underline{n}}}^* - E(a_{\underline{n}} | \underline{r}_{\underline{n}}; H_{\underline{n}}) | K_{\underline{n}}) = \\ &= P\left(\frac{\underline{v}_{\underline{n}} - E(\underline{v}_{\underline{n}} | K_{\underline{n}})}{\sigma(\underline{v}_{\underline{n}} | K_{\underline{n}})} \geq k_{n,r_{\underline{n}}}^* \frac{\sigma(a_{\underline{n}} | \underline{r}_{\underline{n}}; H_{\underline{n}})}{\sigma(\underline{v}_{\underline{n}} | K_{\underline{n}})} - \frac{E(\underline{v}_{\underline{n}} | K_{\underline{n}})}{\sigma(\underline{v}_{\underline{n}} | K_{\underline{n}})} | K_{\underline{n}}\right) \rightarrow 1 - \Phi(u_{\alpha} - \eta) \end{aligned}$$

by application of a well-known theorem (see e.g. [6], theorem 20.6 or [9], Korollar 2.6).

REMARK. Lemma 2.2 is a particular case of lemma 2.3.

THEOREM 2.1.

(a) Let $p_2 \in (0,1)$ and $b > 0$ be fixed, and let $p_{1,n} = p_2 + b/\sqrt{n}$. If n_1 and n_2 , the respective numbers of experiments A and B, satisfy $n_1 + n_2 = n$, $\lim_{n \rightarrow \infty} n_1/n = \zeta \in [0,1]$ and $n_1, n_2 \rightarrow \infty$, then the unconditional power $\beta_{\alpha,n}^{(1)}$ of Fisher's test at level α at the alternative $(p_{1,n}, p_2)$ satisfies

$$(2.8) \quad \lim_{n \rightarrow \infty} \beta_{\alpha, n}^{(1)} = 1 - \Phi(u_{\alpha} - \delta_1 b)$$

where

$$(2.9) \quad \delta_1 = \sqrt{\frac{\zeta(1-\zeta)}{p_2(1-p_2)}}$$

(b) Let $p_2 \in (0,1)$ and $b > 0$ be fixed, and let $p_{1,n} = cp_2 + b/\sqrt{n}$. If method II is applied with a binomially- (n, π) distributed number $\underline{n}_1$ of experiments A, and $\underline{n}_2 = n - \underline{n}_1$ experiments B, then the unconditional power $\beta_{\alpha, n}^{(2)}$ of method II for testing " $p_1 = cp_2$ " at level α at the alternative $(p_{1,n}, p_2)$ satisfies

$$(2.10) \quad \lim_{n \rightarrow \infty} \beta_{\alpha, n}^{(2)} = 1 - \Phi(u_{\alpha} - \delta_2 b)$$

where

$$(2.11) \quad \delta_2 = \sqrt{\frac{\pi(1-\pi)}{(\pi c + 1 - \pi)cp_2}}$$

(c) Let $p_2 \in (0,1)$ and $b > 0$ be fixed, and let $p_{1,n} = cp_2 + d + b/\sqrt{n}$. The (unconditional) power $\beta_{\alpha, n}^{(3)}$ of method III for testing " $p_1 = cp_2 + d$ " at level α at the alternative $(p_{1,n}, p_2)$ satisfies

$$(2.12) \quad \lim_{n \rightarrow \infty} \beta_{\alpha, n}^{(3)} = 1 - \Phi(u_{\alpha} - \delta_3 b)$$

where

$$(2.13) \quad \delta_3 = \sqrt{\frac{1}{(d+c)(1-d)}}$$

PROOF. (a) Let $k_{n,r}^{(1)}$ denote the critical value of the Fisher test at level α . Then (see e.g. [9], Satz 2.11)

¹⁾ Notice, that n_1 and n_2 are functions of n , so $k_{n,r}$ and $k_{n,r}^*$ (see next page) depend on n_1 and n_2 through n only.

$$k_{n,r}^* \stackrel{\text{def}}{=} \frac{k_{n,r} - \frac{n_1 r}{n}}{\sqrt{\frac{n_1 n_2 r (n-r)}{n^2 (n-1)}}} \rightarrow u_\alpha$$

for any sequence of hypergeometric- (n, n_1, r) distributions, provided that $r, n-r, n_1, n_2 \rightarrow \infty$, which is a sufficient condition for asymptotic normality (see e.g. [7]). So, if a random variable $\underline{r}$ satisfies $\underline{r}/n \rightarrow c \in (0,1)$ in probability, then $k_{n,\underline{r}}^* \rightarrow u_\alpha$ in probability. Now let

$$H \stackrel{\text{def}}{\longleftrightarrow} p_1 = p_2$$

$$K_n \stackrel{\text{def}}{\longleftrightarrow} p_1 = p_2 + b/\sqrt{n}$$

$$\underline{v}_n \stackrel{\text{def}}{=} \underline{s}_1 - E(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) = (n_2 \underline{s}_1 - n_1 \underline{s}_2)/n$$

Then

$$E(\underline{v}_n \mid K_n) = \frac{n_2 n_1 (p_2 + b/\sqrt{n}) - n_1 n_2 p_2}{n} = \frac{n_1 n_2 b}{n\sqrt{n}}$$

$$\sigma^2(\underline{v}_n \mid K_n) = \frac{n_2^2}{n^2} n_1 \left(p_2 + \frac{b}{\sqrt{n}}\right) \left(1 - p_2 - \frac{b}{\sqrt{n}}\right) + \frac{n_1^2}{n^2} n_2 p_2 (1 - p_2)$$

Since conditions (2.3) to (2.5) with $\underline{a}_n = \underline{s}_1$ and $\underline{r}_n = \underline{s}_1 + \underline{s}_2$ are obviously fulfilled, lemma 2.3 can be applied:

$$\lim_{n \rightarrow \infty} \frac{E(\underline{v}_n \mid K_n)}{\sigma(\underline{v}_n \mid K_n)} = \sqrt{\frac{\zeta(1-\zeta)}{p_2(1-p_2)}} b \quad \square$$

REMARK. A stronger result is obtained by ALBERS [1], for the case $0 < \zeta < 1$ (example 2.3.1).

(b) Let k_r denote the conditional critical value for testing $p_1 = cp_2$ against $p_1 > cp_2$ at level α . Then, if $r \rightarrow \infty$, with $\theta \stackrel{\text{def}}{=} \pi c / (\pi c + 1 - \pi)$

$$k_{n,r}^* \stackrel{\text{def}}{=} \frac{k_r - \theta r}{\sqrt{r\theta(1-\theta)}} \rightarrow u_\alpha$$

(see e.g. [9], Satz 2.11).

Now let

$$H \stackrel{\text{def}}{\longleftrightarrow} p_1 = cp_2$$

$$K_n \stackrel{\text{def}}{\longleftrightarrow} p_1 = cp_2 + b_2/\sqrt{n}$$

$$\begin{aligned} \underline{v}_n &= \underline{s}_1 - E(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) = \\ &= \underline{s}_1 - \theta(\underline{s}_1 + \underline{s}_2) = \frac{(1-\pi)\underline{s}_1 - \pi c \underline{s}_2}{\pi c + 1 - \pi} \end{aligned}$$

$\underline{s}_1$ and $\underline{s}_2$ are dependent, but $(\underline{s}_1, \underline{s}_2)$ is asymptotically bivariate normal, so $\underline{v}_n$ is asymptotically normal. Further

$$\begin{aligned} \sigma^2(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) &= (\underline{s}_1 + \underline{s}_2)\theta(1-\theta) \\ \sigma^2(\underline{v}_n \mid K_n) &= E\sigma^2(\underline{v}_n \mid \underline{s}_1 + \underline{s}_2; K_n) + \sigma^2 E(\underline{v}_n \mid \underline{s}_1 + \underline{s}_2; K_n) = \\ &= E(\underline{s}_1 + \underline{s}_2)\theta_n(1-\theta_n) + \sigma^2((\theta_n - \theta)(\underline{s}_1 + \underline{s}_2)) \\ &\quad \text{(with } \theta_n = \frac{\pi p_1}{\pi p_1 + (1-\pi)p_2} \text{ under } K_n) \\ &= n v_n \theta_n (1 - \theta_n) + (\theta_n - \theta)^2 n v_n (1 - v_n) \\ &\quad \text{(with } v_n = \pi p_1 + (1-\pi)p_2 \text{ under } K_n) \end{aligned}$$

So, with $\underline{a}_n = \underline{s}_1$ and $\underline{r}_n = \underline{s}_1 + \underline{s}_2$, the conditions (2.3) and (2.4) are obviously fulfilled and lemma 2.3 can be applied.

$$\begin{aligned} E(\underline{v}_n \mid K_n) &= \frac{(1-\pi)\pi(cp_2 + b_2/\sqrt{n})n - \pi c(1-\pi)p_2 n}{\pi c + 1 - \pi} \\ &= \frac{(1-\pi)\pi b \sqrt{n}}{\pi c + 1 - \pi} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{E(\underline{v}_n | K_n)}{\sigma(\underline{v}_n | K_n)} = \frac{(1-\pi)\pi b}{(\pi c + 1 - \pi)\sqrt{v\theta(1-\theta)}}$$

$$(\text{with } v = (\pi c + 1 - \pi)p_2)$$

$$= \sqrt{\frac{\pi(1-\pi)}{(\pi c + 1 - \pi)cp_2}} b \quad \square$$

(c) The statistic $\underline{s}_1 + \underline{f}_2$ has a bin- (n, θ_n) distribution with

$$\theta_n = \theta = \frac{p_1}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{cp_2+d}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{d+c}{1+c} \quad \text{under the hypothesis}$$

and

$$\theta_n = \frac{p_1}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{d+c}{1+c} + \frac{b}{(1+c)\sqrt{n}} \quad \text{under the alternative}$$

By lemma 2.2 we obtain the asymptotic power:

$$1 - \Phi\left(u_\alpha - \frac{\frac{b}{1+c}}{\sqrt{\frac{d+c}{1+c} \frac{1-d}{1+c}}}\right) = 1 - \Phi(u_\alpha - \delta_3 b). \quad \square$$

The values of δ_i (see (2.9), (2.11) and (2.13)) immediately lead to the Pitman-efficiencies, which are, according to the definition as given by WITTING and NÖLLE [9], equal to δ_i^2/δ_j^2 . Therefore we have

$$(2.14) \quad \text{ARE}(I, II) = \frac{\zeta(1-\zeta)}{\pi(1-\pi)(1-p_2)}$$

$$(2.15) \quad \text{ARE}(I, III) = \frac{\zeta(1-\zeta)}{p_2(1-p_2)}$$

$$(2.16) \quad \text{ARE}(II, III) = \frac{\pi(1-\pi)}{(\pi c + 1 - \pi)p_2}$$

From (2.9) it is clear that for Fisher's test $\zeta = \frac{1}{2}$ is the best choice. From (2.11) it can easily be derived that $\pi = 1/(1+\sqrt{c})$ is asymptotically optimal for method II. With these values for ζ and π the efficiencies are

$$(2.17) \quad \text{ARE(I,II)} = \frac{1}{1-p_2}$$

$$(2.18) \quad \text{ARE(I,III)} = \frac{1}{4p_2(1-p_2)}$$

$$(2.19) \quad \text{ARE(II,III)} = \frac{1}{p_2(1+\sqrt{c})^2}$$

These results are the same as the results that were presumed and confirmed by numerical investigations in [3]. Formulae (2.17) and (2.18) lead to the conclusion that Fisher's test is at least as good as the other methods for testing $p_1 = p_2$. Only if $p_2 \approx \frac{1}{2}$, methods I and II are equally good, but this is of little importance, since usually $p_2 \approx \frac{1}{2}$ cannot be guaranteed in advance. If $p_1 = cp_2$ ($c \neq 1$) must be tested, method III is better than method II, except if $p_2 < (1+\sqrt{c})^{-2}$. If $p_1 - cp_2 = d$ ($d \neq 0$) must be tested, only method III can be used.

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